Energy Structure of Optimal Positional Strategies in Mean Payoff Games

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Abstract

This note studies structural aspects concerning Optimal Positional Strategies (OPSs) in Mean Payoff Games (MPGs), it's a contribution to understanding the relationship between OPSs in MPGs and Small Energy-Progress Measures (SEPMs) in reweighted Energy Games (EGs). Firstly, it is observed that the space of all OPSs, $\texttt{opt}_\Gamma\Sigma^M_0$, admits a *unique complete decomposition* in terms of so-called *extremal*-SEPMs in reweighted EGs; this points out what we called the "Energy-Lattice $\mathscr{X}^*_\Gamma$ of $\texttt{opt}_\Gamma\Sigma^M_0$". Secondly, it is offered a *pseudo-polynomial total-time* recursive procedure for *enumerating* (w/o repetitions) all the elements of $\mathscr{X}^*_\Gamma$, and for computing the corresponding partitioning of $\texttt{opt}_\Gamma\Sigma^M_0$. It is observed that the corresponding recursion tree defines an additional lattice $\mathscr{B}^*_\Gamma$, whose elements are certain subgames $\Gamma' \subseteq \Gamma$ that we call *basic* subgames. The extremal-SEPMs of a given MPG Γ coincide with the least-SEPMs of the basic subgames of Γ; so, $\mathscr{X}^*_\Gamma$ is the energy-lattice comprising all and only the *least*-SEPMs of the *basic* subgames of Γ. The complexity of the proposed enumeration for both $\mathscr{B}^*_\Gamma$ and $\mathscr{X}^*_\Gamma$ is $O(|V|^3|E|W|\mathscr{B}^*_\Gamma|)$ total time and $O(|V||E|) + \Theta\big(|E||\mathscr{B}^*_\Gamma|\big)$ working space. Finally, it is constructed an MPG Γ for which $|\mathscr{B}^*_\Gamma| > |\mathscr{X}^*_\Gamma|$, this proves that $\mathscr{B}^*_\Gamma$ and $\mathscr{X}^*_\Gamma$ are not isomorphic.

Keywords: Mean Payoff Games, Optimal Strategy Synthesis, Pseudo-Polynomial Time, Energy Games, Small Energy-Progress Measures.

1. Introduction

A *Mean Payoff Game* (MPG) is a two-player infinite game $\Gamma = (V, E, w, \langle V_0, V_1\rangle)$, that is played on a finite weighted directed graph, denoted $G^\Gamma \triangleq (V, E, w)$, where $w : E \to \mathbb{Z}$, the vertices of which are partitioned into two classes, V_0 and V_1, according to the player to which they belong.

At the beginning of the game a pebble is placed on some vertex $v_s \in V$, then the two players, named Player 0 and Player 1, move it along the arcs ad infinitum. Assuming the pebble is currently on some $v \in V_0$, then Player 0 chooses an arc $(v, v') \in E$ going out of v and moves the pebble to the destination vertex v'. Similarly, if the pebble is currently on some $v \in V_1$, it is Player 1's turn to choose an outgoing arc. The infinite sequence $v_s, v, v' \ldots$ of all the encountered vertices forms a *play*. In order to play well, Player 0 wants to maximize the limit inferior of the long-run average weight of the traversed arcs, i.e., to maximize $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} w(v_i, v_{i+1})$, whereas Player 1 wants to minimize the $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} w(v_i, v_{i+1})$. Ehrenfeucht and Mycielski (1979) proved that each vertex v admits a *value*, denoted $\texttt{val}^\Gamma(v)$, that each player can secure by means of a *memoryless* (or *positional*) strategy, i.e., one depending only on the current vertex position and not on the previous choices.

Preprint submitted to arXiv *Received: date / Accepted: date*

Solving an MPG consists in computing the values of all vertices (*Value Problem*) and, for each player, a positional strategy that secures such values to that player (*Optimal Strategy Synthesis*). The corresponding decision problem lies in $\mathsf{NP}\cap\mathsf{coNP}$ (Zwick and Paterson, 1996) and it was later shown to be in $\mathsf{UP}\cap\mathsf{coUP}$ (Jurdziński, 1998).

The problem of devising efficient algorithms for solving MPGs has been studied extensively in the literature. The first milestone was settled in Gurvich et al. (1988), in which it was offered an *exponential* time algorithm for solving a slightly wider class of MPGs called *Cyclic Games*. Afterwards, Zwick and Paterson (1996) devised the first deterministic procedure for computing values in MPGs, and optimal strategies securing them, within a pseudo-polynomial time and polynomial space. In particular, it was established an $O(|V|^3|E|W)$ upper bound for the time complexity of the Value Problem, as well as an upper bound of $O(|V|^4|E|W\log(|E|/|V|))$ for that of Optimal Strategy Synthesis (Zwick and Paterson, 1996).

Several research efforts have been spent in studying quantitative extensions of infinite games for modeling quantitative aspects of reactive systems, e.g., the *Energy Games (EGs)* (Chakrabarti et al., 2003; Bouyer et al., 2008; Brim et al., 2011). These studies unveiled interesting connections between EGs and MPGs; and by relying on these techniques, recently the worst-cast time complexity of the Value Problem and Optimal Strategy Synthesis was given an improved pseudo-polynomial upper bound (Comin and Rizzi, 2015, 2016a); those works focused on offering a simple proof of the improved upper bound. However, the running time of the proposed algorithm turned out to be also $\Omega(|V|^2|E|W)$, the actual time complexity being $\Theta\big(|V|^2|E|W + \sum_{v\in V}\mathtt{deg}_\Gamma(v)\cdot\ell^0_\Gamma(v)\big)$, where $\ell^0_\Gamma(v)\leq(|V|-1)|V|W$ denotes the total number of times that a certain energy-lifting operator $\delta(\cdot,v)$ is applied to any $v\in V$. A way to overcome this issue was found in Comin and Rizzi (2016b), where a novel algorithmic scheme, named *Jumping*, was introduced; by tackling on some further regularities of the problem, the estimate on the pseudo-polynomial time complexity of MPGs was reduced to: $O(|E|\log|V|)+\Theta\big(\sum_{v\in V}\mathtt{deg}_\Gamma(v)\cdot\ell^1_\Gamma(v)\big)$, where, for every $v\in V$, $\ell^1_\Gamma(v)$ is the total number of applications of $\delta(\cdot,v)$ that are made by the algorithm; $\ell^1_\Gamma\leq(|V|-1)|V|W$ (worst-case, but experimentally $\ell^1_\Gamma\ll\ell^0_\Gamma$; see Comin and Rizzi (2016b)), and the working space is $\Theta(|V|+|E|)$. With this, the pseudo-polynomiality was confined to depend solely on the total number ℓ^1_Γ of required energy-liftings.

Contribution. This work studies the relationship between Optimal Positional Strategies (OPSs) in MPGs and Small Energy-Progress Measures (SEPMs) in reweighted EGs. Actually this paper is an extended and revised version of Section 5 in Comin and Rizzi (2015). Here, we offer:

1. *An Energy-Lattice Decomposition of the Space of Optimal Positional Strategies in MPGs.*

Let's denote by $\mathtt{opt}_\Gamma\Sigma^M_0$ the space of all the optimal positional strategies in a given MPG Γ. What allows the algorithms given in Comin and Rizzi (2015, 2016a,b) to compute at least one $\sigma^*_0\in\mathtt{opt}_\Gamma\Sigma^M_0$ is a *compatibility* relation that links optimal arcs in MPGs to arcs that are *compatible* w.r.t. least-SEPMs in reweighted EGs. The family $\mathscr{E}_\Gamma$ of all SEPMs of a given EG Γ forms a complete finite lattice, the Energy-Lattice of the EG Γ. Firstly, we observe that even though compatibility w.r.t. *least*-SEPMs in reweighted EGs implies optimality of positional strategies in MPGs (see Theorem 3), the converse doesn't hold generally (see Proposition 5). Thus a natural question was whether compatibility w.r.t. SEPMs was really appropriate to capture (e.g., to provide a recursive enumeration of) the whole $\mathtt{opt}_\Gamma\Sigma^M_0$ and not just a proper subset of it. Partially motivated by this question we explored on the relationship between $\mathtt{opt}_\Gamma\Sigma^M_0$ and $\mathscr{E}_\Gamma$. In Theorem 4, it is observed a unique complete decomposition of $\mathtt{opt}_\Gamma\Sigma^M_0$ which is expressed in terms of so called *extremal*-SEPMs in reweighted EGs. This points out what we called the

"Energy-Lattice $\mathscr{X}^*_\Gamma$ associated to $\mathtt{opt}_\Gamma\Sigma^M_0$", the family of all the extremal-SEPMs of a given MPG Γ. So, compatibility w.r.t. SEPMs actually turns out to be appropriate for constructing the whole $\mathtt{opt}_\Gamma\Sigma^M_0$; but an entire lattice $\mathscr{X}^*_\Gamma$ of extremal-SEPMs then arises (and not just the least-SEPM, which turns out to account only for the join/top component of $\mathtt{opt}_\Gamma\Sigma^M_0$).

2. *A Recursive Enumeration of Extremal-SEPMs and Optimal Positional Strategies in MPGs.*

It is offered a pseudo-polynomial total time recursive procedure for enumerating (w/o repetitions) all the elements of $\mathscr{X}^*_\Gamma$, and for computing the associated partitioning of $\mathtt{opt}_\Gamma\Sigma^M_0$. This shows that the above mentioned compatibility relation is appropriate so to extend the algorithm given in Comin and Rizzi (2016b), recursively, in order to compute the whole $\mathtt{opt}_\Gamma\Sigma^M_0$ and $\mathscr{X}^*_\Gamma$. It is observed that the corresponding recursion tree actually defines an additional lattice $\mathscr{B}^*_\Gamma$, whose elements are certain subgames $\Gamma' \subseteq \Gamma$ that we call *basic* subgames. The extremal-SEPMs of a given Γ coincide with the least-SEPMs of the basic subgames of Γ; so, $\mathscr{X}^*_\Gamma$ is the energy-lattice comprising all and only the *least*-SEPMs of the *basic* subgames of Γ. The total time complexity of the proposed enumeration for both $\mathscr{B}^*_\Gamma$ and $\mathscr{X}^*_\Gamma$ is $O(|V|^3|E|W|\mathscr{B}^*_\Gamma|)$, it works in space $O(|V||E|)+\Theta\big(|E||\mathscr{B}^*_\Gamma|\big)$. An example of MPG Γ for which $|\mathscr{B}^*_\Gamma| > |\mathscr{X}^*_\Gamma|$ ends this paper.

Organization. The following Section 2 introduces some notation and provides the required background on infinite 2-player pebble games and related algorithmic results. In Section 3, a suitable relation between values, optimal strategies, and certain reweighting operations is recalled from Comin and Rizzi (2015, 2016a). Section 4 offers a unique and complete energy-lattice decomposition of $\mathtt{opt}_\Gamma\Sigma^M_0$. Finally, Section 5 provides a recursive enumeration of $\mathscr{X}^*_\Gamma$ and the corresponding partitioning of $\mathtt{opt}_\Gamma\Sigma^M_0$.

2. Notation and Preliminaries

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ the set of natural, integer, and rational numbers. It will be sufficient to consider integral intervals, e.g., $[a,b] \triangleq \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ and $[a,b) \triangleq \{z \in \mathbb{Z} \mid a \leq z < b\}$ for any $a,b \in \mathbb{Z}$. Our graphs are directed and weighted on the arcs; thus, if $G = (V,E,w)$ is a graph, then every arc $e \in E$ is a triplet $e = (u,v,w_e)$, where $w_e = w(u,v) \in \mathbb{Z}$. Let $W \triangleq \max_{e\in E} |w_e|$ be the maximum absolute weight. Given a vertex $u \in V$, the set of its successors is $N^{\text{out}}_\Gamma(u) \triangleq \{v \in V \mid (u,v) \in E\}$, and the set of its predecessors is $N^{\text{in}}_\Gamma(u) \triangleq \{v \in V \mid (v,u) \in E\}$. Let $\mathtt{deg}_\Gamma(v) \triangleq |N^{\text{in}}_\Gamma(v)| + |N^{\text{out}}_\Gamma(v)|$. A *path* is a sequence $v_0v_1\ldots v_n\ldots$ such that $\forall^{i\in[n]}\,(v_i,v_{i+1}) \in E$. Let V^* be the set of all (possibly empty) finite paths. A *simple path* is a finite path $v_0v_1\ldots v_n$ having no repetitions, i.e., for any $i,j \in [0,n]$ it holds $v_i \neq v_j$ if $i \neq j$. A *cycle* is a path $v_0v_1\ldots v_{n-1}v_n$ such that $v_0\ldots v_{n-1}$ is simple and $v_n = v_0$. The *average weight* of a cycle $v_0\ldots v_n$ is $w(C)/|C| = \frac{1}{n}\sum_{i=0}^{n-1} w(v_i,v_{i+1})$. A cycle $C = v_0v_1\ldots v_n$ is *reachable* from v in G if there is some path p in G such that $p \cap C \neq \emptyset$.

An *arena* is a tuple $\Gamma = (V,E,w,\langle V_0,V_1\rangle)$ where $G^\Gamma \triangleq (V,E,w)$ is a finite weighted directed graph and (V_0,V_1) is a partition of V into the set V_0 of vertices owned by Player 0, and V_1 owned by Player 1. It is assumed that G^Γ has no sink, i.e., $\forall^{v\in V} N^{\text{out}}_\Gamma(v) \neq \emptyset$; we remark that G^Γ is not required to be a bipartite graph on colour classes V_0 and V_1. A subarena Γ' (or *subgame*) of Γ is any arena $\Gamma' = (V',E',w',\langle V'_0,V'_1\rangle)$ such that: $V' \subseteq V$, $\forall^{i\in\{0,1\}} V'_i = V' \cap V_i$, $E' \subseteq E$, and $\forall^{e\in E'} w'_e = w_e$. Given $S \subseteq V$, the subarena of Γ induced by S is denoted $\Gamma_{|S}$, its vertex set is S and its edge set is $E' = \{(u,v) \in E \mid u,v \in S\}$. A game on Γ is played for infinitely many rounds by two players moving a pebble along the arcs of G^Γ. At the beginning of the game the pebble

is found on some vertex $v_s \in V$, which is called the *starting position* of the game. At each turn, assuming the pebble is currently on a vertex $v \in V_i$ (for $i = 0, 1$), Player i chooses an arc $(v, v') \in E$ and then the next turn starts with the pebble on v'. Below, Fig. 1 depicts an example arena Γ_{ex}.

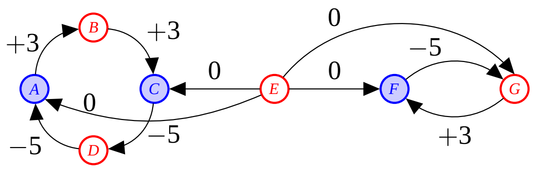


Figure 1: An arena $\Gamma_{\text{ex}} = \langle V, \mathsf{E}, w, (V_0, V_1)\rangle$. Here, $V = \{A, B, C, D, E, F, G\}$ and $\mathsf{E} = \{(A, B, +3), (B, C, +3), (C, D, -5), (D, A, -5), (E, A, 0), (E, C, 0), (E, F, 0), (E, G, 0), (F, G, -5), (G, F, +3)\}$. Also, $V_0 = \{B, D, E, G\}$ is colored in red, while $V_1 = \{A, C, F\}$ is filled in blue.

A *play* is any infinite path $v_0 v_1 \dots v_n \dots \in V^\omega$ in Γ. For any $i \in \{0, 1\}$, a strategy of Player i is any function $\sigma_i : V^* \times V_i \to V$ such that for every finite path $p'v$ in G^Γ, where $p' \in V^*$ and $v \in V_i$, it holds that $(v, \sigma_i(p', v)) \in E$. A strategy σ_i of Player i is *positional* (or *memoryless*) if $\sigma_i(p, v_n) = \sigma_i(p', v'_m)$ for every finite paths $pv_n = v_0 \dots v_{n-1} v_n$ and $p'v'_m = v'_0 \dots v'_{m-1} v'_m$ in G^Γ such that $v_n = v'_m \in V_i$. The set of all the positional strategies of Player i is denoted by Σ_i^M. A play $v_0 v_1 \dots v_n \dots$ is *consistent* with a strategy $\sigma \in \Sigma_i$ if $v_{j+1} = \sigma(v_0 v_1 \dots v_j)$ whenever $v_j \in V_i$.

Given a starting position $v_s \in V$, the *outcome* of two strategies $\sigma_0 \in \Sigma_0$ and $\sigma_1 \in \Sigma_1$, denoted $\mathtt{outcome}^\Gamma(v_s, \sigma_0, \sigma_1)$, is the unique play that starts at v_s and is consistent with both σ_0 and σ_1.

Given a memoryless strategy $\sigma_i \in \Sigma_i^M$ of Player i in Γ, then $G(\sigma_i, \Gamma) = (V, E_{\sigma_i}, w)$ is the graph obtained from G^Γ by removing all the arcs $(v, v') \in E$ such that $v \in V_i$ and $v' \neq \sigma_i(v)$; we say that $G(\sigma_i, \Gamma)$ is obtained from G^Γ *by projection* w.r.t. σ_i.

For any weight function $w' : E \to \mathbb{Z}$, the *reweighting* of $\Gamma = (V, E, w, \langle V_0, V_1\rangle)$ w.r.t. w' is the arena $\Gamma^{w'} = (V, E, w', \langle V_0, V_1\rangle)$. Also, for $w : E \to \mathbb{Z}$ and any $v \in \mathbb{Z}$, we denote by $w + v$ the weight function w' defined as $\forall^{e \in E} w'_e \triangleq w_e + v$. Indeed, we shall consider reweighted games of the form Γ^{w-q}, for some $q \in \mathbb{Q}$. Notice that the corresponding weight function $w' : E \to \mathbb{Q} : e \mapsto w_e - q$ is rational, while we required the weights of the arcs to be always integers. To overcome this issue, it is sufficient to re-define Γ^{w-q} by scaling all weights by a factor equal to the denominator of $q \in \mathbb{Q}$; i.e., when $q \in \mathbb{Q}$, say $q = N/D$ for $\gcd(N, D) = 1$ we define $\Gamma^{w-q} \triangleq \Gamma^{D \cdot w - N}$. This rescaling operation doesn't change the winning regions of the corresponding games, let's denote this equivalence as $\Gamma^{w-q} \cong \Gamma^{D \cdot w - N}$, and it has the significant advantage of allowing for a discussion (and an algorithmics) which is strictly based on integer weights.

2.1. *Mean Payoff Games*

A *Mean Payoff Game* (MPG) (Brim et al., 2011; Zwick and Paterson, 1996; Ehrenfeucht and Mycielski, 1979) is a game played on some arena Γ for infinitely many rounds by two opponents, Player 0 gains a payoff defined as the long-run average weight of the play, whereas Player 1 loses that value. Formally, the Player 0's *payoff* of a play $v_0 v_1 \dots v_n \dots$ in Γ is defined as follows:

$$\mathtt{MP}_0(v_0 v_1 \dots v_n \dots) \triangleq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} w(v_i, v_{i+1}).$$

The value *secured* by a strategy $\sigma_0 \in \Sigma_0$ in a vertex v is defined as:

$$\mathtt{val}^{\sigma_0}(v) \triangleq \inf_{\sigma_1 \in \Sigma_1} \mathtt{MP}_0\big(\mathtt{outcome}^{\Gamma}(v, \sigma_0, \sigma_1)\big),$$

Notice that payoffs and secured values can be defined symmetrically for the Player 1 (i.e., by interchanging the symbol *0* with *1* and *inf* with *sup*).

Ehrenfeucht and Mycielski Ehrenfeucht and Mycielski (1979) proved that each vertex $v \in V$ admits a unique *value*, denoted $\mathtt{val}^{\Gamma}(v)$, which each player can secure by means of a *memoryless* (or *positional*) strategy. Moreover, *uniform* positional optimal strategies do exist for both players, in the sense that for each player there exist at least one positional strategy which can be used to secure all the optimal values, independently with respect to the starting position v_s. Thus, for every MPG Γ, there exists a strategy $\sigma_0 \in \Sigma_0^M$ such that $\forall^{v \in V} \mathtt{val}^{\sigma_0}(v) \geq \mathtt{val}^{\Gamma}(v)$, and there exists a strategy $\sigma_1 \in \Sigma_1^M$ such that $\forall^{v \in V} \mathtt{val}^{\sigma_1}(v) \leq \mathtt{val}^{\Gamma}(v)$. The *(optimal) value* of a vertex $v \in V$ in the MPG Γ is given by:

$$\mathtt{val}^{\Gamma}(v) = \sup_{\sigma_0 \in \Sigma_0} \mathtt{val}^{\sigma_0}(v) = \inf_{\sigma_1 \in \Sigma_1} \mathtt{val}^{\sigma_1}(v).$$

Thus, a strategy $\sigma_0 \in \Sigma_0$ is *optimal* if $\mathtt{val}^{\sigma_0}(v) = \mathtt{val}^{\Gamma}(v)$ for all $v \in V$. We denote $\mathrm{opt}_{\Gamma}\Sigma_0^M \triangleq \big\{\sigma_0 \in \Sigma_0^M(\Gamma) \mid \forall^{v \in V} \mathtt{val}^{\Gamma}_{\sigma_0}(v) = \mathtt{val}^{\Gamma}(v)\big\}$. A strategy $\sigma_0 \in \Sigma_0$ is said to be *winning* for Player 0 if $\forall^{v \in V} \mathtt{val}^{\sigma_0}(v) \geq 0$, and $\sigma_1 \in \Sigma_1$ is winning for Player 1 if $\mathtt{val}^{\sigma_1}(v) < 0$. Correspondingly, a vertex $v \in V$ is a *winning starting position* for Player 0 if $\mathtt{val}^{\Gamma}(v) \geq 0$, otherwise it is winning for Player 1. The set of all winning starting positions of Player i is denoted by $\mathscr{W}_i$ for $i \in \{0, 1\}$.

A refined formulation of the determinacy theorem is offered in Björklund et al. (2004).

Theorem 1 (Björklund et al. (2004))**.** *Let Γ be an MPG and let $\{C_i\}_{i=1}^m$ be a partition (called* ergodic*) of its vertices into $m \geq 1$ classes each one having the same optimal value $\nu_i \in \mathbb{Q}$. Formally, $V = \bigsqcup_{i=1}^m C_i$ and $\forall^{i \in [m]} \forall^{v \in C_i} \mathtt{val}^{\Gamma_i}(v) = \nu_i$, where $\Gamma_i \triangleq \Gamma_{|C_i}$.*

Then, Player 0 has no vertices with outgoing arcs leading from C_i to C_j whenever $\nu_i < \nu_j$, and Player 1 has no vertices with outgoing arcs leading from C_i to C_j whenever $\nu_i > \nu_j$;

moreover, there exist $\sigma_0 \in \Sigma_0^M$ and $\sigma_1 \in \Sigma_1^M$ such that:

– If the game starts from any vertex in C_i, then σ_0 secures a gain at least ν_i to Player 0 and σ_1 secures a loss at most ν_i to Player 1;

– Any play that starts from C_i always stays in C_i, if it is consistent with both strategies σ_0, σ_1, i.e., if Player 0 plays according to σ_0, and Player 1 according to σ_1.

A finite variant of MPGs is well-known in the literature (Ehrenfeucht and Mycielski, 1979; Zwick and Paterson, 1996; Brim et al., 2011), where the game stops as soon as a cyclic sequence of vertices is traversed. It turns out that this is equivalent to the infinite game formulation (Ehrenfeucht and Mycielski, 1979), in the sense that the values of an MPG are in a strong relationship with the average weights of its cycles, as in the next lemma.

Proposition 1 (Brim, *et al.* Brim et al. (2011))**.** *Let Γ be an MPG. For all $\nu \in \mathbb{Q}$, for all $\sigma_0 \in \Sigma_0^M$, and for all $v \in V$, the value $\mathtt{val}^{\sigma_0}(v)$ is greater than ν iff all cycles C reachable from v in the projection graph $G^{\Gamma}_{\sigma_0}$ have an average weight $w(C)/|C|$ greater than ν.*

The proof of Proposition 1 follows from the memoryless determinacy of MPGs. We remark that a proposition which is symmetric to Proposition 1 holds for Player 1 as well: for all $\nu \in \mathbb{Q}$, for all positional strategies $\sigma_1 \in \Sigma_1^M$ of Player 1, and for all vertices $v \in V$, the value $\mathtt{val}^{\sigma_1}(v)$ is

less than ν *iff* if all cycles reachable from v in the projection graph $G^{\Gamma}_{\sigma_1}$ have an average weight less than ν. Also, it is well-known (Brim et al., 2011; Ehrenfeucht and Mycielski, 1979) that each value $\mathtt{val}^{\Gamma}(v)$ is contained within the following set of rational numbers:

$$S_\Gamma = \Big\{ N/D \mid D \in [1, |V|], N \in [-D \cdot W, D \cdot W] \Big\}.$$

Notice, $|S_\Gamma| \leq |V|^2 W$.

The present work focuses on the algorithmics of the following classical problem:

– *Optimal Strategy Synthesis.* Compute an optimal positional strategy for Player 0 in Γ.

Also, in Section 5 we shall consider the problem of computing the whole $\mathtt{opt}_\Gamma \Sigma^M_0$:

– *Optimal Strategy Enumeration.* Provide a listing[1] of all the optimal positional strategies of Player 0 in the MPG Γ.

2.2. *Energy Games and Small Energy-Progress Measures*

An *Energy Game* (EG) is a game that is played on an arena Γ for infinitely many rounds by two opponents, where the goal of Player 0 is to construct an infinite play $v_0 v_1 \ldots v_n \ldots$ such that for some initial *credit* $c \in \mathbb{N}$ the following holds: $c + \sum_{i=0}^{j} w(v_i, v_{i+1}) \geq 0$, for all $j \geq 0$. Given an initial credit $c \in \mathbb{N}$, a play $v_0 v_1 \ldots v_n \ldots$ is *winning* for Player 0 if it satisfies (1), otherwise it is winning for Player 1. A vertex $v \in V$ is a winning starting position for Player 0 if there exists an initial credit $c \in \mathbb{N}$ and a strategy $\sigma_0 \in \Sigma_0$ such that, for every strategy $\sigma_1 \in \Sigma_1$, the play $\mathtt{outcome}^{\Gamma}(v, \sigma_0, \sigma_1)$ is winning for Player 0. As in the case of MPGs, the EGs are memoryless determined Brim et al. (2011), i.e., for every $v \in V$, either v is winning for Player 0 or v is winning for Player 1, and (uniform) memoryless strategies are sufficient to win the game. In fact, as shown in the next lemma, the decision problems of MPGs and EGs are intimately related.

Proposition 2 (Brim et al. (2011))**.** *Let Γ be an arena. For all threshold $\nu \in \mathbb{Q}$, for all vertices $v \in V$, Player* 0 *has a strategy in the MPG Γ that secures value at least ν from v if and only if, for some initial credit $c \in \mathbb{N}$, Player* 0 *has a winning strategy from v in the reweighted EG $\Gamma^{w-\nu}$.*

In this work we are especially interested in the *Minimum Credit Problem* (MCP) for EGs: for each winning starting position v, compute the minimum initial credit $c^* = c^*(v)$ such that there exists a winning strategy $\sigma_0 \in \Sigma^M_0$ for Player 0 starting from v. A fast pseudo-polynomial time deterministic procedure for solving MCPs comes from Brim et al. (2011).

Theorem 2 (Brim et al. (2011))**.** *There exists a deterministic algorithm for solving the MCP within $O(|V|\,|E|\,W)$ pseudo-polynomial time, on any input EG $(V, E, w, \langle V_0, V_1 \rangle)$.*

The algorithm mentioned in Theorem 2 is the *Value-Iteration* algorithm (Brim et al., 2011). Its rationale relies on the notion of *Small Energy-Progress Measures* (SEPMs).

2.3. *Energy-Lattices of Small Energy-Progress Measures*

Small-Energy Progress Measures are bounded, non-negative and integer-valued functions that impose local conditions to ensure global properties on the arena, in particular, witnessing that Player 0 has a way to enforce conservativity (i.e., non-negativity of cycles) in the resulting

[1] The listing has to be exhaustive (i.e., each element is listed eventually) and without repetitions (i.e., no element is listed twice).

game's graph. Recovering standard notation, see e.g. Brim et al. (2011), let us denote $\mathscr{C}_\Gamma = \{n \in \mathbb{N} \mid n \le (|V|-1)W\} \cup \{\top\}$ and let $\preceq$ be the total order on $\mathscr{C}_\Gamma$ defined as: $x \preceq y$ *iff* either $y = \top$ or $x, y \in \mathbb{N}$ and $x \le y$. In order to cast the minus operation to range over $\mathscr{C}_\Gamma$, let us consider an operator $\ominus : \mathscr{C}_\Gamma \times \mathbb{Z} \to \mathscr{C}_\Gamma$ defined as follows:

$$a \ominus b \triangleq \begin{cases} \max(0, a-b), & \text{if } a \neq \top \text{ and } a-b \le (|V|-1)W; \\ a \ominus b = \top, & \text{otherwise.} \end{cases}$$

Given an EG Γ on vertex set $V = V_0 \cup V_1$, a function $f : V \to \mathscr{C}_\Gamma$ is a *Small Energy-Progress Measure* (SEPM) for Γ if and only if the following two conditions are met:

1. if $v \in V_0$, then $f(v) \succeq f(v') \ominus w(v, v')$ for *some* $(v, v') \in E$;
2. if $v \in V_1$, then $f(v) \succeq f(v') \ominus w(v, v')$ for *all* $(v, v') \in E$.

The values of a SEPM, i.e., the elements of the image $f(V)$, are called the *energy levels* of f. It is worth to denote by $V_f = \{v \in V \mid f(v) \neq \top\}$ the set of vertices having finite energy. Given a SEPM $f : V \to \mathscr{C}_\Gamma$ and a vertex $v \in V_0$, an arc $(v, v') \in E$ is said to be *compatible* with f whenever $f(v) \succeq f(v') \ominus w(v, v')$; otherwise (v, v') is said to be *incompatible* with f. Moreover, a positional strategy $\sigma_0 \in \Sigma_0^M$ is said to be *compatible* with f whenever: $\forall^{v \in V_0}$ if $\sigma_0(v) = v'$ then $(v, v') \in E$ is compatible with f; otherwise, σ_0 is *incompatible* with f.

It is well-known that the family of all the SEPMs of a given Γ forms a complete (finite) lattice, which we denote by $\mathscr{E}_\Gamma$ call it the *Energy-Lattice* of Γ. Therefore, we shall consider:

$$\mathscr{E}_\Gamma \triangleq \big(\{f : V \to \mathscr{C}_\Gamma \mid f \text{ is SEPM of } \Gamma\}, \sqsubseteq\big),$$

where for any two SEPMs f, g define $f \sqsubseteq g$ *iff* $\forall v \in V\, f(v) \preceq g(v)$. Notice that, whenever f and g are SEPMs, then so is the *minimum function* defined as: $\forall^{v \in V} h(v) \triangleq \min\{f(v), g(v)\}$. This fact allows one to consider the *least* SEPM, namely, the unique SEPM $f^* : V \to \mathscr{C}_\Gamma$ such that, for any other SEPM $g : V \to \mathscr{C}_\Gamma$, the following holds: $\forall^{v \in V} f^*(v) \preceq g(v)$. Thus, $\mathscr{E}_\Gamma$ is a complete lattice. So, $\mathscr{E}_\Gamma$ enjoys of *Knaster–Tarski Theorem*, which states that the set of fixed-points of a monotone function on a complete lattice is again a complete lattice.

Also concerning SEPMs, we shall rely on the following lemmata. The first one relates SEPMs to the winning region $\mathscr{W}_0$ of Player 0 in EGs.

Proposition 3 (Brim et al. (2011))**.** *Let Γ be an EG. Then the following hold.*

1. *If f is any SEPM of the EG Γ and $v \in V_f$, then v is a winning starting position for Player 0 in the EG Γ. Stated otherwise, $V_f \subseteq \mathscr{W}_0$;*
2. *If f^* is the least SEPM of the EG Γ, and v is a winning starting position for Player 0 in the EG Γ, then $v \in V_{f^*}$. Thus, $V_{f^*} = \mathscr{W}_0$.*

The following bound holds on the energy-levels of any SEPM (by definition of $\mathscr{C}_\Gamma$).

Proposition 4. *Let Γ be an EG. Let f be any SEPM of Γ.*
Then, for every $v \in V$ either $f(v) = \top$ or $0 \le f(v) \le (|V|-1)W$.

3. Optimal Strategies from Reweightings

It is now recalled a sufficient condition, for a positional strategy to be optimal, which is expressed in terms of reweighted EGs and their SEPMs.

Theorem 3 (Comin and Rizzi (2016a)). *Let $\Gamma = (V, E, w, \langle V_0, V_1 \rangle)$ be an MPG. For each $u \in V$, consider the reweighted EG $\Gamma_u \cong \Gamma^{w - \mathtt{val}^{\Gamma}(u)}$. Let $f_u : V \to \mathscr{C}_{\Gamma_u}$ be any SEPM of Γ_u such that $u \in V_{f_u}$ (i.e., $f_u(u) \neq \top$). Moreover, we assume: $f_{u_1} = f_{u_2}$ whenever $\mathtt{val}^{\Gamma}(u_1) = \mathtt{val}^{\Gamma}(u_2)$.*

When $u \in V_0$, let $v_{f_u} \in N^{out}_{\Gamma}(u)$ be any vertex such that $(u, v_{f_u}) \in E$ is compatible with f_u in EG Γ_u, and consider the positional strategy $\sigma_0^* \in \Sigma_0^M$ defined as follows: $\forall^{u \in V_0}\ \sigma_0^*(u) \triangleq v_{f_u}$.

Then, σ_0^* is an optimal positional strategy for Player 0 in the MPG Γ.

Proof. See the proof of [Theorem 4 in Comin and Rizzi (2016a)]. □

Remark 1. *Notice that Theorem 3 holds, particularly, when f_u is the least SEPM f_u^* of the reweighted EG Γ_u. This is because $u \in V_{f_u^*}$ always holds for the least SEPM f_u^* of the EG Γ_u: indeed, by Proposition 2 and by definition of Γ_u, then u is a winning starting position for Player 0 in the EG Γ_u (for some initial credit); thus, by Proposition 3, it follows that $u \in V_{f_u^*}$.*

4. An Energy-Lattice Decomposition of $\mathtt{opt}_{\Gamma}\Sigma_0^M$

Recall the example arena Γ_{ex} shown in Fig. 1. It is easy to see that $\forall^{v \in V}\,\mathtt{val}^{\Gamma_{\text{ex}}}(v) = -1$. Indeed, Γ_{ex} contains only two cycles, i.e., $C_L = [A, B, C, D]$ and $C_R = [F, G]$, also notice that $w(C_L)/C_L = w(C_R)/C_R = -1$. The least-SEPM f^* of the reweighted EG Γ_{ex}^{w+1} can be computed by running a Value Iteration (Brim et al., 2011). Taking into account the reweighting $w \rightsquigarrow w+1$, as in Fig. 2: $f^*(A) = f^*(E) = f^*(G) = 0$, $f^*(B) = f^*(D) = f^*(F) = 4$, and $f^*(C) = 8$.

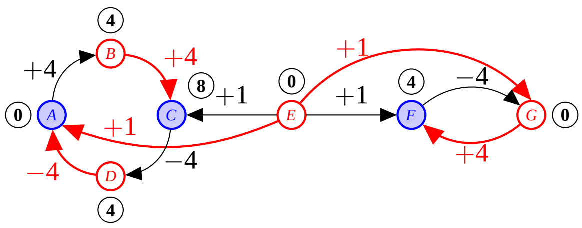


Figure 2: The least-SEPM f^* of Γ_{ex}^{w+1} (energy-levels are depicted in circled boldface). All and only those arcs of Player 0 that are compatible with f^* are $(B,C), (D,A), (E,A), (E,G), (G,F)$ (thick red arcs).

So, Γ_{ex} (Fig. 2) implies the following.

Proposition 5. *The converse statement of Theorem 3 doesn't hold; there exist infinitely many MPGs Γ having at least one $\sigma_0 \in \mathtt{opt}_{\Gamma}\Sigma_0^M$ which is not compatible with the least-SEPM of Γ.*

Proof. Consider the Γ_{ex} of Fig. 2, and the least-SEPM f^* of the EG Γ_{ex}^{w+1}. The only vertex at which Player 0 really has a choice is E. Every arc going out of E is optimal in the MPG Γ_{ex}: whatever arc $(E, X) \in \mathsf{E}$ (for any $X \in \{A, C, F, G\}$) Player 0 chooses at E, the resulting payoff equals $\mathtt{val}^{\Gamma_{\text{ex}}}(E) = -1$. Let f^* be the least-SEPM of f^* in Γ_{ex}^{w+1}. Observe, (E, C) and (E, F) are not compatible with f^* in Γ_{ex}^{w+1}, only (E, A) and (E, G) are. For instance, the positional strategy $\sigma_0 \in \Sigma_0^M$ defined as $\sigma_0(E) \triangleq F$, $\sigma_0(B) \triangleq C$, $\sigma_0(D) \triangleq A$, $\sigma_0(G) \triangleq F$ ensures a payoff $\forall^{v \in V}\,\mathtt{val}^{\Gamma_{\text{ex}}}(v) = -1$, but it is not compatible with the least-SEPM f^* of Γ_{ex}^{w+1} (because $f^*(E) = 0 < 3 = f^*(F) \ominus w(E, F)$). It is easy to turn the Γ_{ex} of Fig. 2 into a family on infinitely

many similar examples. $\square$

We now aim at strengthening the relationship between $\mathrm{opt}_\Gamma\Sigma_0^M$ and the Energy-Lattice $\mathscr{E}_\Gamma$. For this, we assume *wlog* $\exists^{\nu\in\mathbb{Q}}\forall^{v\in V}\mathtt{val}^\Gamma(v)=\nu$; this follows from Theorem 1, which allows one to partition Γ into several domains $\Gamma_i \triangleq \Gamma_{|C_i}$ each one satisfying: $\exists^{\nu_i\in\mathbb{Q}}\forall^{v\in C_i}\mathtt{val}^{\Gamma_i}(v)=\nu_i$. By Theorem 1 we can study $\mathtt{opt}_{\Gamma_i}\Sigma_0^M$, independently w.r.t. $\mathtt{opt}_{\Gamma_j}\Sigma_0^M$ for $j\neq i$.

We say that an MPG Γ is ν-*valued* if and only if $\exists^{\nu\in\mathbb{Q}}\forall^{v\in V}\mathtt{val}^\Gamma(v)=\nu$.

Given an MPG Γ and $\sigma_0\in\Sigma_0^M(\Gamma)$, recall, $G(\Gamma,\sigma_0)\triangleq(V,E',w')$ is obtained from G^Γ by deleting all and only those arcs that are not part of σ_0, i.e.,

$$E'\triangleq\big\{(u,v)\in E \mid u\in V_0 \text{ and } v=\sigma_0(u)\big\}\cup\big\{(u,v)\in E\mid u\in V_1\big\},$$

where each $e\in E'$ is weighted as in Γ, i.e., $w':E'\to\mathbb{Z}:e\mapsto w_e$.

When $G=(V,E,w)$ is a weighted directed graph, a *feasible-potential (FP)* for G is any map $\pi:V\to\mathscr{C}_G$ s.t. $\forall^{u\in V}\forall^{v\in N^{\mathrm{out}}(u)}\pi(u)\succeq\pi(v)\ominus w(u,v)$. The *least*-FP $\pi^*=\pi^*_G$ is the (unique) FP s.t., for any other FP π, it holds $\forall^{v\in V}\pi^*(v)\preceq\pi(v)$. Given G, the Bellman-Ford algorithm can be used to produce π^*_G in $O(|V||E|)$ time. Let $\pi^*_{G(\Gamma,\sigma_0)}$ be the *least-FP* of $G(\Gamma,\sigma_0)$. Notice, for every $\sigma_0\in\Sigma_0^M$, the least-FP $\pi^*_{G(\Gamma,\sigma_0)}$ is actually a SEPM for the EG Γ; still it can differ from the least-SEPM of Γ, due to σ_0. We consider the following family of strategies.

Definition 1 ($\Delta_0^M(f,\Gamma)$-Strategies)**.** *Let $\Gamma=\langle V,E,w,(V_0,V_1)\rangle$ and let $f:V\to\mathscr{C}_\Gamma$ be a SEPM for the EG Γ. Let $\Delta_0^M(f,\Gamma)\subseteq\Sigma_0^M(\Gamma)$ be the family of all and only those positional strategies of Player 0 in Γ s.t. $\pi^*_{G(\Gamma,\sigma_0)}$ coincides with f pointwisely, i.e.,*

$$\Delta_0^M(f,\Gamma)\triangleq\Big\{\sigma_0\in\Sigma_0^M(\Gamma)\,|\,\forall^{v\in V}\,\pi^*_{G(\Gamma,\sigma_0)}(v)=f(v)\Big\}.$$

We now aim at exploring further on the relationship between $\mathscr{E}_\Gamma$ and $\mathtt{opt}_\Gamma\Sigma_0^M$, via $\Delta_0^M(f,\Gamma)$.

Definition 2 (The Energy-Lattice of $\mathtt{opt}_\Gamma\Sigma_0^M$)**.** *Let Γ be a ν-valued MPG. Let $\mathscr{X}\subseteq\mathscr{E}_{\Gamma^{w-\nu}}$ be a sublattice of SEPMs of the reweighted EG $\Gamma^{w-\nu}$.*

We say that $\mathscr{X}$ is an "Energy-Lattice *of $\mathtt{opt}_\Gamma\Sigma_0^M$" iff $\forall^{f\in\mathscr{X}}\Delta_0^M(f,\Gamma^{w-\nu})\neq\emptyset$ and the following disjoint-set decomposition holds:*

$$\mathtt{opt}_\Gamma\Sigma_0^M=\bigsqcup_{f\in\mathscr{X}}\Delta_0^M(f,\Gamma^{w-\nu}).$$

Lemma 1. *Let Γ be a ν-valued MPG, and let $\sigma_0^*\in\mathtt{opt}_\Gamma\Sigma_0^M$. Then, $G(\Gamma^{w-\nu},\sigma_0^*)$ is* conservative *(i.e., it contains* no *negative cycle).*

Proof. Let $C\triangleq(v_1\ldots,v_k,v_1)$ by any cycle in $G(\Gamma^{w-\nu},\sigma_0^*)$. Since we have $\sigma_0^*\in\mathtt{opt}_\Gamma\Sigma_0^M$ and $\forall^{v\in V}\mathtt{val}^\Gamma(v)=\nu$, thus $w(C)/k=\frac{1}{k}\sum_{i=1}^k w(v_i,v_{i+1})\geq\nu$ (for $v_{k+1}\triangleq v_1$) by Proposition 1, so that, assuming $w'\triangleq w-\nu$, then: $w'(C)/k=\frac{1}{k}\sum_{i=1}^k\big(w(v_i,v_{i+1})-\nu\big)=w(C)/k-\nu\geq\nu-\nu=0$. $\square$

Some aspects of the following Proposition 6 rely heavily on Theorem 3: the compatibility relation comes again into play. Moreover, we observe that Proposition 6 is equivalent to the

following fact, which provides a sufficient condition for a positional strategy to be optimal. Consider a ν-valued MPG Γ, for some $\nu \in \mathbb{Q}$, and let $\sigma_0^* \in \mathtt{opt}_\Gamma \Sigma_0^M$. Let $\hat{\sigma}_0 \in \Sigma_0^M(\Gamma)$ be any (not necessarily optimal) positional strategy for Player 0 in the MPG Γ. Suppose the following holds:

$$\forall^{v \in V}\, \pi^*_{G(\Gamma^{w-\nu}, \hat{\sigma}_0)}(v) = \pi^*_{G(\Gamma^{w-\nu}, \sigma_0^*)}(v).$$

Then, by Proposition 6, $\hat{\sigma}_0$ is an optimal positional strategy for Player 0 in the MPG Γ.

We are thus relying on the same *compatibility* relation between Σ_0^M and SEPMs in reweighted EGs which was at the *base* of Theorem 3, aiming at extending Theorem 3 so to describe the whole $\mathtt{opt}_\Gamma \Sigma_0^M$ (and not just the join/top component of it).

Proposition 6. *Let the MPG Γ be ν-valued, for some $\nu \in \mathbb{Q}$.*
There is at least one Energy-Lattice of $opt_\Gamma \Sigma_0^M$:

$$\mathscr{X}^*_\Gamma \triangleq \{\pi^*_{G(\Gamma^{w-\nu}, \sigma_0)} \mid \sigma_0 \in opt_\Gamma \Sigma_0^M\}.$$

Proof. The only non-trivial point to check being: $\bigsqcup_{f \in \mathscr{X}^*_\Gamma} \Delta_0^M(f, \Gamma^{w-\nu}) \subseteq \mathtt{opt}_\Gamma \Sigma_0^M$.

For this, we shall rely on Theorem 3. Let $\hat{f} \in \mathscr{X}^*_\Gamma$ and $\hat{\sigma}_0 \in \Delta_0^M(\hat{f}, \Gamma^{w-\nu})$ be fixed (arbitrarily). Since $\hat{f} \in \mathscr{X}^*_\Gamma$, then $\hat{f} = \pi^*_{G(\Gamma^{w-\nu}, \sigma_0^*)}$ for some $\sigma_0^* \in \mathtt{opt}_\Gamma \Sigma_0^M$. Therefore, the following holds:

$$\pi^*_{G(\Gamma^{w-\nu}, \hat{\sigma}_0)} = \hat{f} = \pi^*_{G(\Gamma^{w-\nu}, \sigma_0^*)}.$$

Clearly, $\hat{\sigma}_0$ is compatible with $\hat{f}$ in the EG $\Gamma^{w-\nu}$, because $\hat{f} = \pi^*_{G(\Gamma^{w-\nu}, \hat{\sigma}_0)}$. By Lemma 1, since σ_0^* is optimal, then $G(\Gamma^{w-\nu}, \sigma_0^*)$ is conservative. Therefore:

$$V_{\hat{f}} = V_{\pi^*_{G(\Gamma^{w-\nu}, \sigma_0^*)}} = V.$$

Notice, $\hat{\sigma}_0$ satisfies exactly the hypotheses required by Theorem 3. Therefore, $\hat{\sigma}_0 \in \mathtt{opt}_\Gamma \Sigma_0^M$. This proves (*).This also shows $\mathtt{opt}_\Gamma \Sigma_0^M = \bigsqcup_{f \in \mathscr{X}^*_\Gamma} \Delta_0^M(f, \Gamma^{w-\nu})$, and concludes the proof. □

Proposition 7. *Let the MPG Γ be ν-valued, for some $\nu \in \mathbb{Q}$. Let ${\mathscr{X}^*_\Gamma}_1$ and ${\mathscr{X}^*_\Gamma}_2$ be two Energy-Lattices for $opt_\Gamma \Sigma_0^M$. Then, ${\mathscr{X}^*_\Gamma}_1 = {\mathscr{X}^*_\Gamma}_2$.*

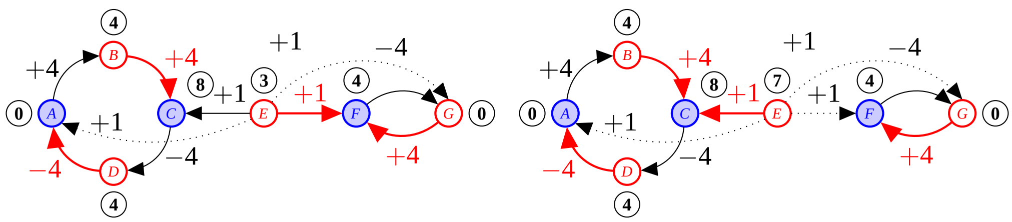
Proof. By symmetry, it is sufficient to prove that ${\mathscr{X}^*_\Gamma}_1 \subseteq {\mathscr{X}^*_\Gamma}_2$. Let $f_1 \in {\mathscr{X}^*_\Gamma}_1$ be fixed (arbitrarily). Then, $f_1 = \pi^*_{G(\Gamma^{w-\nu}, \hat{\sigma}_0)}$ for some $\hat{\sigma}_0 \in \mathtt{opt}_\Gamma \Sigma_0^M$. Since $\hat{\sigma}_0 \in \mathtt{opt}_\Gamma \Sigma_0^M$ and since ${\mathscr{X}^*_\Gamma}_2$ is an Energy-Lattices, there exists $f_2 \in {\mathscr{X}^*_\Gamma}_2$ s.t. $\hat{\sigma}_0 \in \Delta_0^M(f_2, \Gamma^{w-\nu})$, which implies $\pi^*_{G(\Gamma^{w-\nu}, \hat{\sigma}_0)} = f_2$. Thus, $f_1 = \pi^*_{G(\Gamma^{w-\nu}, \hat{\sigma}_0)} = f_2$. This implies $f_1 \in {\mathscr{X}^*_\Gamma}_2$. □

The next theorem summarizes the main point of this section.

Theorem 4. *Let Γ be a ν-valued MPG, for some $\nu \in \mathbb{Q}$. Then, $\mathscr{X}^*_\Gamma \triangleq \{\pi^*_{G(\Gamma^{w-\nu}, \sigma_0)} \mid \sigma_0 \in opt_\Gamma \Sigma_0^M\}$ is the unique Energy-Lattice of $opt_\Gamma \Sigma_0^M$.*

Proof. By Proposition 6 and Proposition 7. □

Example 1. *Consider the MPG Γ_{ex}, as defined in Fig. 1. Then, $\mathscr{X}^*_{\Gamma_{ex}} = \{f^*, f_1, f_2\}$, where f^* is the least-SEPM of the reweighted EG Γ^{w+1}_{ex}, and where the following holds: $f_1(A) = f_2(A) = f^*(A) = 0$; $f_1(B) = f_2(B) = f^*(B) = 4$; $f_1(C) = f_2(C) = f^*(C) = 8$; $f_1(D) = f_2(D) = f^*(D) = 4$; $f_1(F) = f_2(F) = f^*(F) = 4$; $f_1(G) = f_2(G) = f^*(G) = 0$; finally, $f^*(E) = 0$, $f_1(E) = 3$, $f_2(E) = 7$. An illustration of f_1 is offered in Fig. 3a (energy-levels are depicted in circled boldface). whereas f_2 is depicted in Fig. 3b. Notice that $f^*(v) \leq f_1(v) \leq f_2(v)$ for every $v \in V$, and this ordering relation is illustrated in Fig. 3.*



(a) The extremal-SEPM f_1 of Γ^{w+1}_{ex} (b) The extremal-SEPM f_2 of Γ^{w+1}_{ex}.

Definition 3. *Each element $f \in \mathscr{X}^*_\Gamma$ is called* extremal-*SEPM.*

The next lemma is the converse of Lemma 1.

Lemma 2. *Let the MPG Γ be ν-valued, for some $\nu \in \mathbb{Q}$. Consider any $\sigma_0 \in \Sigma^M_0(\Gamma)$, and assume that $G(\Gamma^{w-\nu}, \sigma_0)$ is conservative. Then, $\sigma_0 \in \mathtt{opt}_\Gamma\Sigma^M_0$.*

Proof. Let $C = (v_1, \ldots, v_\ell v_1)$ any cycle in $G(\Gamma, \sigma_0)$. Then, the following holds (if $v_{\ell+1} = v_1$): $\frac{w(C)}{\ell} = \frac{1}{\ell}\sum_{i=1}^{\ell} w(v_i, v_{i+1}) = \nu + \frac{1}{\ell}\sum_{i=1}^{\ell}\Big(w(v_i, v_{i+1}) - \nu\Big) \geq \nu$, where $\frac{1}{\ell}\sum_{i=1}^{\ell}\big(w(v_i, v_{i+1}) - \nu\big) \geq 0$ holds because $G(\Gamma^{w-\nu}, \sigma_0)$ is conservative. By Proposition 1, since $w(C)/\ell \geq \nu$ for every cycle C in $G^\Gamma_{\sigma_0}$, then $\sigma_0 \in \mathtt{opt}_\Gamma\Sigma^M_0$. □

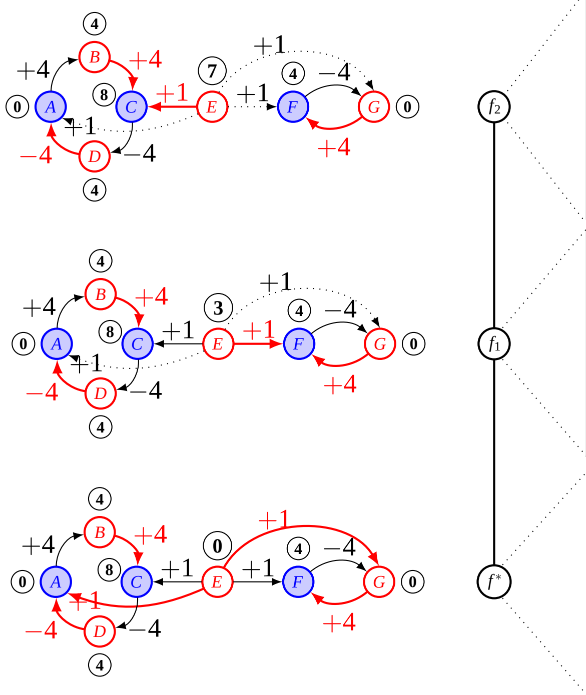
The following proposition asserts some properties of the extremal-SEPMs.

Proposition 8. *Let the MPG Γ be ν-valued, for some $\nu \in \mathbb{Q}$. Let $\mathscr{X}^*_\Gamma$ be the Energy-Lattice of $\mathtt{opt}_\Gamma\Sigma^M_0$. Moreover, let $f : V \to \mathscr{C}_\Gamma$ be a SEPM for the reweighted EG $\Gamma^{w-\nu}$. Then, the following three properties are equivalent:*

1. $f \in \mathscr{X}^*_\Gamma$;
2. *There exists $\sigma_0 \in \mathtt{opt}_\Gamma\Sigma^M_0$ s.t. $\pi^*_{G(\Gamma^{w-\nu}, \sigma_0)}(v) = f(v)$ for every $v \in V$.*
3. $V_f = \mathscr{W}_0(\Gamma^{w-\nu}) = V$ *and* $\Delta^M_0(f, \Gamma^{w-\nu}) \neq \emptyset$;

Proof of (1 $\iff$ 2). Indeed, $\mathscr{X}^*_\Gamma = \{\pi^*_{G(\Gamma^{w-\nu}, \sigma_0)} \mid \sigma_0 \in \mathtt{opt}_\Gamma\Sigma^M_0\}$. □

Proof of (1 $\Rightarrow$ 3). Assume $f \in \mathscr{X}^*_\Gamma$. Since (1 $\iff$ 2), there exist $\sigma_0 \in \mathtt{opt}_\Gamma\Sigma^M_0$ s.t. $\pi^*_{G(\Gamma^{w-\nu}, \sigma_0)} = f$. Thus, $\sigma_0 \in \Delta^M_0(f, \Gamma^{w-\nu})$, so that $\Delta^M_0(f, \Gamma^{w-\nu}) \neq \emptyset$. We claim $V_f = \mathscr{W}_0(\Gamma^{w-\nu}) = V$. Since $\forall (v \in V)\, \mathtt{val}^\Gamma(v) = \nu$, then $\mathscr{W}_0(\Gamma^{w-\nu}) = V$ by Proposition 2. Next, $G(\Gamma^{w-\nu}, \sigma_0)$ is conservative



$\Delta_0^M(f_1,\Gamma^{w-\nu}) = \{\sigma_0^{(4)}\}$
$\sigma_0^{(4)}(B) = C$
$\sigma_0^{(4)}(D) = A$
$\sigma_0^{(4)}(E) = C$
$\sigma_0^{(4)}(G) = F$

$\Delta_0^M(f_1,\Gamma^{w-\nu}) = \{\sigma_0^{(3)}\}$
$\sigma_0^{(3)}(B) = C$
$\sigma_0^{(3)}(D) = A$
$\sigma_0^{(3)}(E) = F$
$\sigma_0^{(3)}(G) = F$

$\Delta_0^M(f^*,\Gamma^{w-\nu}) = \{\sigma_0^{(1)}, \sigma_0^{(2)}\}$	
$\sigma_0^{(1)}(B) = C$	$\sigma_0^{(2)}(B) = C$
$\sigma_0^{(1)}(D) = A$	$\sigma_0^{(2)}(D) = A$
$\sigma_0^{(1)}(E) = A$	$\sigma_0^{(2)}(E) = G$
$\sigma_0^{(1)}(G) = F$	$\sigma_0^{(2)}(G) = F$

Figure 3: The decomposition of $\texttt{opt}_\Gamma\Sigma_0^M$ (right), for the MPG Γ_{ex}, which corresponds to the Energy-Lattice $\mathscr{X}^*_{\Gamma_{\text{ex}}} = \{f^*, f_1, f_2\}$ (center) (as in Example 1). Here, $f^* \preceq f_1 \preceq f_2$. This brings a lattice $\mathscr{D}^*_{\Gamma_{\text{ex}}}$ of 3 basic subgames of Γ_{ex} (left).

by Lemma 1. Since $G(\Gamma^{w-\nu}, \sigma_0)$ is conservative and $f = \pi^*_{G(\Gamma^{w-\nu},\sigma_0)}$, then $V_f = V$. Therefore, $V_f = \mathscr{W}_0(\Gamma^{w-\nu}) = V$. $\square$

Proof of ($1 \Leftarrow 3$). Since $\Delta_0^M(f, \Gamma^{w-\nu}) \neq \emptyset$, pick some $\sigma_0 \in \Delta_0^M(f, \Gamma^{w-\nu})$; so, $f = \pi^*_{G(\Gamma^{w-\nu},\sigma_0)}$. Since $V_f = V$ and $f = \pi^*_{G(\Gamma^{w-\nu},\sigma_0)}$, then $G(\Gamma^{w-\nu}, \sigma_0)$ is conservative. Since $G(\Gamma^{w-\nu}, \sigma_0)$ is conservative, then $\sigma_0 \in \texttt{opt}_\Gamma\Sigma_0^M$ by Lemma 2. Since $f = \pi^*_{G^*}$ and $\sigma_0 \in \texttt{opt}_\Gamma\Sigma_0^M$, then $f \in \mathscr{X}^*_\Gamma$ because $2 \Rightarrow 1$. $\square$

5. A Recursive Enumeration of $\mathscr{X}^*_\Gamma$ and $\texttt{opt}_\Gamma(\Sigma_0^M)$

An enumeration algorithm for a set S provides an exhaustive listing of all the elements of S (without repetitions). As mentioned in Section 4, by Theorem 1, no loss of generality occurs if we assume Γ to be ν-valued for some $\nu \in \mathbb{Q}$. One run of the algorithm given in Comin and Rizzi (2016b) allows one to partition an MPG Γ, into several domains Γ_i each one being ν_i-valued for $\nu_i \in S_\Gamma$; in $O(|V|^2|E|W)$ time and linear space. Still, by Proposition 5, Theorem 3 is not sufficient for enumerating the whole $\texttt{opt}_\Gamma(\Sigma_0^M)$; it is enough only for $\Delta_0^M(f^*_\nu, \Gamma^{w-\nu})$ where f^*_ν is

the least-SEPM of $\Gamma^{w-\nu}$, which is just the join/top component of $\texttt{opt}_\Gamma(\Sigma_0^M)$. However, thanks to Theorem 4, we now have a refined description of $\texttt{opt}_\Gamma \Sigma_0^M$ in terms $\mathscr{X}_\Gamma^*$.

We offer a recursive enumeration of all the extremal-SEPMs, i.e., $\mathscr{X}_\Gamma^*$, and for computing the corresponding partitioning of $\texttt{opt}_\Gamma\big(\Sigma_0^M\big)$. In order to avoid duplicate elements in the enumeration, the algorithm needs to store a lattice $\mathscr{B}_\Gamma^*$ of subgames of Γ, which is related to $\mathscr{X}_\Gamma^*$. We assume to have a data-structure T_Γ supporting the following operations, given a subarena Γ' of Γ: $\texttt{insert}(\Gamma', T_\Gamma)$ stores Γ' into T_Γ; $\texttt{contains}(\Gamma', T_\Gamma)$ returns $\texttt{T}$ if and only if Γ' is in T_Γ, and $\texttt{F}$ otherwise. A simple implementation of T_Γ goes by indexing $N_{\Gamma'}^{\text{out}}(v)$ for each $v \in V$ (e.g., with a trie data-structure). This can run in $O(|E|\log|V|)$ time, consuming $O(|E|)$ space per stored item. Similarly, one can index SEPMs in $O(|V|\log(|V|W))$ time and $O(|V|)$ space per stored item.

The listing procedure is named `enum()`, it takes a ν-valued MPG Γ and goes as follows.

1. Compute the least-SEPM f^* of Γ, and `print` Γ to output. Theorem 3 can be employed at this stage for enumerating $\Delta_0^M(f^*, \Gamma^{w-\nu})$: indeed, these are all and only those positional strategies lying in the *Cartesian* product of all the arcs $(u,v) \in E$ that are *compatible* with f^* in $\Gamma^{w-\nu}$ (because f^* is the least-SEPM of Γ).
2. Let $\texttt{St} \leftarrow \emptyset$ be an empty stack of vertices.
3. For each $\hat{u} \in V_0$, do the following:
   - Compute $E_{\hat{u}} \leftarrow \{(\hat{u}, v) \in E \mid f^*(\hat{u}) \prec f^*(v) \ominus (w(\hat{u}, v) - \nu)\}$;
   - If $E_{\hat{u}} \neq \emptyset$, then:
     - Let $E' \leftarrow E_{\hat{u}} \cup \{(u,v) \in E \mid u \neq \hat{u}\}$ and $\Gamma' \leftarrow (V, E', w, \langle V_0, V_1\rangle)$.
     - If $\texttt{contains}(\Gamma', T_\Gamma) = \texttt{F}$, do the following:
       * Compute the least-SEPM f'^* of $\Gamma'^{w-\nu}$;
       * If $V_{f'^*} = V$:
         – Push $\hat{u}$ on top of `St` and $\texttt{insert}(\Gamma', T_\Gamma)$.
         – If $\texttt{contains}(f'^*, T_\Gamma) = \texttt{F}$, then $\texttt{insert}(f'^*, T_\Gamma)$ and `print` f'^*.
4. While $\texttt{St} \neq \emptyset$:
   - `pop` $\hat{u}$ from `St`; Let $E_{\hat{u}} \leftarrow \{(\hat{u}, v) \in E \mid f^*(\hat{u}) \prec f^*(v) \ominus (w(\hat{u}, v) - \nu)\}$, and $E' \leftarrow E_{\hat{u}} \cup \{(u,v) \in E \mid u \neq \hat{u}\}$, and $\Gamma' \leftarrow (V, E', w, \langle V_0, V_1\rangle)$;
   - Make a recursive call to `enum()` on input Γ'.

Down the recursion tree, when computing least-SEPMs, the children Value-Iterations can amortize by starting from the energy-levels of the parent. The lattice of subgames $\mathscr{B}_\Gamma^*$ comprises all and only those subgames $\Gamma' \subseteq \Gamma$ that are eventually inserted into T_Γ at Step (3) of `enum`(); these are called the *basic subgames* of Γ. The correctness of `enum()` follows by Theorem 4 and Theorem 3. In summary, we obtain the following result.

Theorem 5. *There exists a recursive algorithm for enumerating (w/o repetitions) all elements of $\mathscr{B}_\Gamma^*$ with time-delay[2] $O(|V|^3|E|W)$, on any input MPG Γ; moreover, the algorithm works with $O(|V||E|) + \Theta\big(|E||\mathscr{B}_\Gamma^*|\big)$ space. So, it enumerates $\mathscr{X}_\Gamma^*$ (w/o repetitions) in $O\big(|V|^3|E|W|\mathscr{B}_\Gamma^*|\big)$ total time, and $O(|V||E|) + \Theta\big(|E||\mathscr{B}_\Gamma^*|\big)$ space.*

[2] A listing algorithm has $O(f(n))$ *time-delay* when the time spent between any two consecutives is $O(f(n))$.

To conclude we observe that $\mathscr{B}^*_\Gamma$ and $\mathscr{X}^*_\Gamma$ are not isomorphic as lattices, not even as sets (the cardinality of $\mathscr{B}^*_\Gamma$ can be greater that that of $\mathscr{X}^*_\Gamma$). Indeed, there is a surjective antitone mapping φ_Γ from $\mathscr{B}^*_\Gamma$ onto $\mathscr{X}^*_\Gamma$, (i.e., φ_Γ sends $\Gamma' \in \mathscr{B}^*_\Gamma$ to its least-SEPM $f^*_{\Gamma'} \in \mathscr{X}^*_\Gamma$); still, we can construct instances of MPGs such that $|\mathscr{B}^*_\Gamma| > |\mathscr{X}^*_\Gamma|$, i.e., φ_Γ is not into and $\mathscr{B}^*_\Gamma$, $\mathscr{X}^*_\Gamma$ are not isomorphic. That would be a case of *degeneracy*, and an example MPG Γ_{d} is given in Fig. 4.

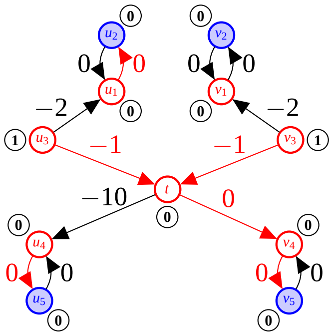


Figure 4: An MPG Γ_{d} for which $|\mathscr{B}^*_\Gamma| > |\mathscr{X}^*_\Gamma|$.

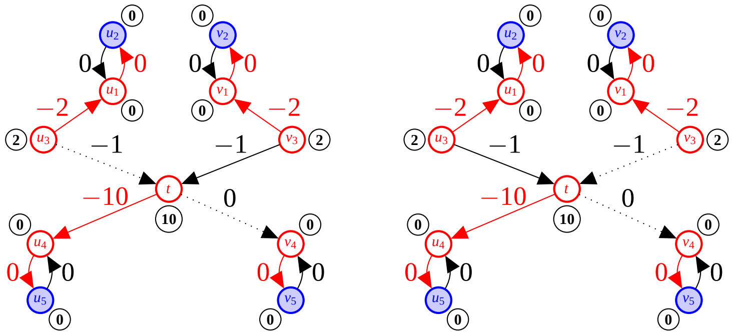


Figure 5: Two basic subgames $\Gamma^1_d \neq \Gamma^2_d$ of Γ_{d}, having the same least-SEPM $f^*_1 = f^*_2$.

In the MPG Γ_{d}, Player 0 has to decide how to move only at u_3, v_3 and t; the remaining moves are forced. The least-SEPM f^* of Γ_{d} is: $f^*(u_3) = 1$, $f^*(v_3) = 1$, $f^*(t) = 0$, and $\forall_{x \in V_{\Gamma_{\mathrm{d}}} \setminus \{u_3,v_3,t\}}\, f^*(x) = 0$; leading to the following memory-less strategy: $\sigma^*_0(u_3) = t$, $\sigma^*_0(v_3) = t$, $\sigma^*_0(t) = v_4$. Then, consider the lattice of subgames $\mathscr{B}^*_{\Gamma_{\mathrm{d}}}$; particularly, consider the following two basic subgames Γ^1_d, Γ^2_d: let Γ'_d be the arena obtained by removing the arc (t, v_4) from Γ_d; let Γ^1_d be the arena obtained by removing the arc (u_3, t) from Γ'_d; let Γ^2_d be the arena obtained by removing the arc (v_3, t) from Γ'_d. See Fig. 5 for an illustration. Next, let f^*_1, f^*_2 be the least-SEPMs of Γ^1_d and Γ^2_d, respectively; then, $f^*_1(u_3) = f^*_2(u_3) = 2$, $f^*_1(v_3) = f^*_2(v_3) = 2$, $f^*_1(t) = f^*_2(t) = 10$, and $\forall_{x \in V_{\Gamma_{\mathrm{d}}} \setminus \{u_3,v_3,t\}}\, f^*_1(x) = f^*_2(x) = 0$. Thus, $\Gamma^1_d \neq \Gamma^2_d$, but $f^*_1 = f^*_2$; this proves that Γ_{d} is *degenerate* and that $\mathscr{B}^*_\Gamma$, $\mathscr{X}^*_\Gamma$ are not isomorphic.

6. Conclusion

We observed a unique complete decomposition of $\texttt{opt}_{\Gamma}\Sigma_0^M$ in terms of extremal-SEPMs in reweighted EGs, also offering a pseudo-polynomial total-time recursive algorithm for enumerating (w/o repetitions) all the elements of $\mathscr{X}_{\Gamma}^*$, i.e., all extremal-SEPMs, and for computing the components of the corresponding partitioning $\mathscr{B}_{\Gamma}^*$ of $\texttt{opt}_{\Gamma}\Sigma_0^M$.

It would be interesting to study further properties enjoyed by $\mathscr{B}_{\Gamma}^*$ and $\mathscr{X}_{\Gamma}^*$; and we ask for more efficient algorithms for enumerating $\mathscr{X}_{\Gamma}^*$, e.g., pseudo-polynomial time-delay and *polynomial space* enumerations.

Acknowledgments. This work was supported by *Department of Computer Science, University of Verona, Verona, Italy*, under PhD grant "Computational Mathematics and Biology", on a co-tutelle agreement with *LIGM, Université Paris-Est in Marne-la-Vallée, Paris, France*.

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